Chemical Evolution of the Universe
Problem sheet 3

1. (a) Consider a gas cloud consisting only of ^{1}H, ^{4}He and ^{7}Li. Given the abundance of ^{7}Li relative to ^{1}H by number, $\frac{n_{\mathrm{Li}}}{n_{\mathrm{H}}} = 10^{-6}$, and the abundance of ^{4}He by mass, $Y = \frac{\text{mass of He}}{\text{total mass}} = 0.25$, what is the abundance of ^{4}He relative to ^{1}H by number, $y = \frac{n_{\mathrm{He}}}{n_{\mathrm{H}}}$?

   (b) Show that Y and y are related by

$$Y = \frac{4y(1-Z)}{1+4y},$$

   where $Z = \frac{\text{mass of metals}}{\text{total mass}}$ is the abundance by mass of all metals.

3 points

2. ^{4}He is used in a range of medical, industrial and scientific applications. So its price actually matters to our everyday lives. Estimate the price (in US$) of a cubic metre of ^{4}He if the neutron lifetime were 100 s instead of 882 s.

   Hint 1: Assume a linear relationship between the price of ^{4}He and its availability (by mass).
   Hint 2: `https://minerals.usgs.gov/minerals/pubs/commodity/helium/mcs-2017-heliu.pdf`

2 points

3. In problem 3. of problem sheet 2 we learned that the small baryon-to-photon ration, η, is the reason that the Universe has to cool to a temperature significantly below that corresponding to the binding energy of deuterium in order for deuterium formation to set in. We will now derive the same result in a different way.

   (a) First, recall from the lecture the formula for the number density of a non-relativistic particle species. Write down the number densities of neutrons and protons, n_n and n_p, remembering that in some places you can use $m_n \approx m_p \approx m_{\mathrm{b}}$ (but not in others).

   (b) Now consider a nucleus X of mass number A and charge Z. Write down its number density n_{X}, remembering that in some places you can use $m_{\mathrm{X}} \approx A m_{\mathrm{b}}$ (but not in others).

   (c) The chemical potential is conserved in the reactions that produces nucleus X from Z protons and $(A-Z)$ neutrons, such that $\mu_{\mathrm{X}} = Z\mu_p + (A-Z)\mu_n$. Use this fact to eliminate μ_{X} from your result in (b).

   (d) Finally, show that

$$\frac{n_{\mathrm{X}}}{n_{\mathrm{b}}} = f(A) \left(\frac{kT}{m_{\mathrm{b}}\, c^2}\right)^{3/2(A-1)} \eta^{A-1} \left(\frac{n_p}{n_{\mathrm{b}}}\right)^{Z} \left(\frac{n_n}{n_{\mathrm{b}}}\right)^{A-Z} \exp\left(\frac{\mathrm{BE}_{\mathrm{X}}}{kT}\right)$$

   where $f(A)$ is a dimensionless number of order a few that depends only on A and g_{X}, and BE_{X} is the binding energy of nucleus X. Remember that $n_{\mathrm{b}} = \eta\, n_\gamma$, and recall the result of problem 3.(b) on problem sheet 2.

   Can you now see why kT has to drop significantly below BE_{X} before the abundance of X can rise to appreciable levels?

8 points