numerical radiative transfer I

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Overview

- The problem
- Numerical solution
- Examples
- Outlook

The Problem

- radiation transport in moving media
- $\rightarrow$ co-moving frame approach
- advantage: all opacities simple to compute
- $\rightarrow$ fast & easy
- but: additional terms in the RTE

Radiative Transfer

$$e\frac{\partial I}{\partial r} + \frac{\partial}{\partial \mu}(fI) + g\frac{\partial}{\partial \lambda}(\lambda I) + hI = \eta - \chi I$$

with

$$\begin{aligned}
e(r,\mu) &= \gamma(\mu+\beta) \\
f(r,\mu) &= \gamma(1-\mu^2)\left[\frac{1+\beta\mu}{r} - \gamma^2(\mu+\beta)\frac{\partial\beta}{\partial r}\right] \\
g(r,\mu) &= \gamma\left[\frac{\beta(1-\mu^2)}{r} + \gamma^2\mu(\mu+\beta)\frac{\partial\beta}{\partial r}\right] \\
h(r,\mu) &= \gamma\left[\frac{\beta(1-\mu^2)}{r} + \gamma^2(1+\mu^2+2\beta\mu)\frac{\partial\beta}{\partial r}\right]
\end{aligned}$$

Radiative Transfer

- $I(r, \mu, \lambda)$: specific intensity scaled by r^2,
- r: radial coordinate,
- μ: cosine of the direction angle, $\mu = \cos\phi$
- v: velocity, $\beta = v/c$, $\gamma^2 = 1/(1 - \beta^2)$,
- $\chi(r, \lambda)$: extinction coefficient, $\chi = \kappa + \sigma_e + \kappa_l \varphi(\lambda)$
- $\eta(r, \lambda)$: emissivity.

Radiative Transfer

Example for $\eta(r, \lambda)$

$$\eta = \kappa B_\lambda(T) + \sigma_e J(\lambda) + \kappa_l \varphi(\lambda) \int_0^\infty \varphi(\lambda) J(\lambda)\, d\lambda$$

with

$$J(\lambda) = \int_{-1}^{1} I(\lambda)\, d\mu$$

- $\kappa B_\lambda(T)$: thermal emission
- $\sigma_e J(\lambda)$: electron scattering
- $\frac{\sigma}{2} \int_0^\infty \int_{-1}^1 \varphi(\lambda) I\, d\mu\, d\lambda$: spectral line emissivity

Numerical solution

- recap: scattering $\rightarrow$

$$S = (1-\epsilon)J + \epsilon B$$

$$\epsilon = \frac{\kappa}{\kappa + \sigma}$$

  - S depends on B and (unknown) J
  - $\rightarrow$ self-consistent solution for J required
  - direct solution expensive
  - use iterative method
  - eigenvalues of iteration matrix close to unity $\rightarrow$ use operator splitting to reduce eigenvalues of amplification matrix

Operator Splitting in RT

- Formal solution (how? see below!)

$$J = \Lambda S$$

- Λ-iteration:

$$\bar{J}_{\rm new} = \Lambda S_{\rm old}, \quad S_{\rm new} = (1-\epsilon)\bar{J}_{\rm new} + \epsilon B$$

- does *not* work for $\tau >> 1$ & small ϵ

Operator Splitting in RT

- Solution: Split Λ operator

$$\Lambda = \Lambda^* + (\Lambda - \Lambda^*)$$

- $\rightarrow$ iteration procedure

$$\bar{J}_{\mathrm{new}} = \Lambda^* S_{\mathrm{new}} + (\Lambda - \Lambda^*) S_{\mathrm{old}}$$

Operator Splitting in RT

- using

$$S = (1-\epsilon)J + \epsilon B$$

and

$$\bar{J}_{\rm fs} = \Lambda S_{\rm old}$$

- gives

$$\left[1-\Lambda^*(1-\epsilon)\right]\bar{J}_{\rm new} = \bar{J}_{\rm fs} - \Lambda^*(1-\epsilon)\bar{J}_{\rm old}$$

Operator Splitting in RT

- solved directly to obtain $\bar{J}_{\rm new}$

$$\bar{J}_{\rm new} = \left[1 - \Lambda^*(1-\epsilon)\right]^{-1} \left(\bar{J}_{\rm fs} - \Lambda^*(1-\epsilon)\bar{J}_{\rm old}\right)$$

Operator Splitting in RT

- mathematically:
- same family as Jacobi or Gauss-Seidel methods
- general form

$$Mx^{k+1} = Nx^k + b$$

for solution of linear system

$$Ax = b$$

with

$$A = M - N$$

Operator Splitting in RT

- operator splitting $\rightarrow$

$$M = 1 - \Lambda^*(1-\epsilon)$$

and

$$N = (\Lambda - \Lambda^*)(1-\epsilon)$$

for the system matrix

$$A = 1 - \Lambda(1-\epsilon)$$

Operator Splitting in RT

- convergence of the iterations $\rightarrow$
- spectral radius $\rho(G) < 1$
- with amplification matrix

$$G = M^{-1}N$$

Operator Splitting in RT

- for this to help
  - eigenvalues of amplification matrix $G \ll 1$
  - works best if $\Lambda^* = \Lambda$ (direct solution, how? see below!)
  - $\rightarrow$ expensive (?)
  - diagonal $\Lambda^* \rightarrow$
    simple but slow convergence
  - band-matrix $\Lambda^* \rightarrow$
    rapid convergence, harder to construct

Operator Splitting in RT

- many possible ways to construct Λ^*
- best: elements of Λ itself
  - no estimates/free parameters
  - 'easy' to compute & use
  - build band-matrix Λ^*

Formal Solution

- pp RTE:

$$\mu \frac{dI}{d\tau_r} = I - S$$

- discretize τ_r (spatial) space
- discretize μ (angle) space
- light path $\rightarrow$ lines of constant μ

Formal Solution

- $\rightarrow$ characteristics of the RTE
- along a characteristic

$$\frac{dI}{d\tau} = I - S$$

- where $I = I(\mu)$ and $\tau = \tau_r/\mu$ are now measured along the characteristic

Formal Solution

- spherical geometry:

$$\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = \chi(S - I)$$

- characteristics = light paths
- static medium → straight lines
- parameterized by *impact parameter* p

$$\mu(r) = \pm\sqrt{1 - p^2/r^2}$$

gives μ for each point along the characteristic p

Formal Solution

- $\rightarrow$ along a characteristic μ varies in spherical geometry
- but the RTE along the characteristic looks like

$$\frac{dI}{d\tau} = I - S$$

- where $I = I(\mu(p))$ and τ are now measured along the characteristic

Formal Solution

- $\rightarrow$ spherical and pp cases reduced to FS along the characteristic
- J computed by numerical quadrature
- need to compute I along the characteristics
- basic idea:
  - approximate S along the characteristic by piece-wise linear or parabolic functions
  - analytically solve FS along the characteristic for I

Formal Solution

- this scheme gives

$$
\begin{aligned}
I(\tau_i) &= I(\tau_{i-1}) \exp(\tau_{i-1} - \tau_i) + \int_{\tau_{i-1}}^{\tau_i} S(\tau) \exp(\tau - \tau_i)\, d\tau \\
I(\tau_i) &\equiv I_{i-1} \exp(-\Delta\tau_{i-1}) + \Delta I_i
\end{aligned}
$$

- τ_i denotes the optical depth along the ray with $\tau_1 \equiv 0$ and τ_{i-1}
- τ is calculated using piecewise linear interpolation of χ along the ray

$$\Delta\tau_{i-1} = (\chi_{i-1} + \chi_i)|s_{i-1} - s_i|/2$$

Formal Solution

- $S(\tau)$ along a characteristic is interpolated by linear or parabolic polynomials so that

$$\Delta I_i = \alpha_i S_{i-1} + \beta_i S_i + \gamma_i S_{i+1}$$

with

$$\begin{aligned}
\alpha_i &= e_{0i} + [e_{2i} - (\Delta\tau_i + 2\Delta\tau_{i-1})e_{1i}]/[\Delta\tau_{i-1}(\Delta\tau_i + \Delta\tau_{i-1})] \\
\beta_i &= [(\Delta\tau_i + \Delta\tau_{i-1})e_{1i} - e_{2i}]/[\Delta\tau_{i-1}\Delta\tau_i] \\
\gamma_i &= [e_{2i} - \Delta\tau_{i-1}e_{1i}]/[\Delta\tau_i(\Delta\tau_i + \Delta\tau_{i-1})]
\end{aligned}$$

for parabolic interpolation and

Formal Solution

$$\begin{aligned}\alpha_i &= e_{0i} - e_{1i}/\Delta\tau_{i-1} \\ \beta_i &= e_{1i}/\Delta\tau_{i-1} \\ \gamma_i &= 0\end{aligned}$$

for linear interpolation.

- auxiliary functions:

$$\begin{aligned}e_{0i} &= 1 - \exp(-\Delta\tau_{i-1}) \\ e_{1i} &= \Delta\tau_{i-1} - e_{0i} \\ e_{2i} &= (\Delta\tau_{i-1})^2 - 2e_{1i}\end{aligned}$$

Formal Solution

- $\Delta\tau_i \equiv \tau_{i+1} - \tau_i$ is the optical depth along the characteristic
- must use linear coefficients at the last integration point along each ray
- some times linear may be better than parabolic (why?)

Formal Solution

- spherical case:
  - tangent rays: FS starts at point 2 with I_1 given as the outer BC and proceeds along the ray
  - FS for 'core-intersecting' ray: split into two parts:
    1. integration from point 1 to point N, where I_1 is given as the outer boundary condition and
    2. integration from point $N+2$ to point $2N$, where I_{N+1} is given as the inner boundary condition.
- pp case: like 'core intersecting' spherical case

Formal Solution

- with this procedure, the I's can be computed for given S
- for all characteristics $\rightarrow$ full RF is known
- could be used for Λ iteration for $\epsilon \approx 1$
- next step: devise procedure to compute Λ^*!
- space-discretized $\rightarrow$
  Λ operator $\rightarrow$ Λ matrix

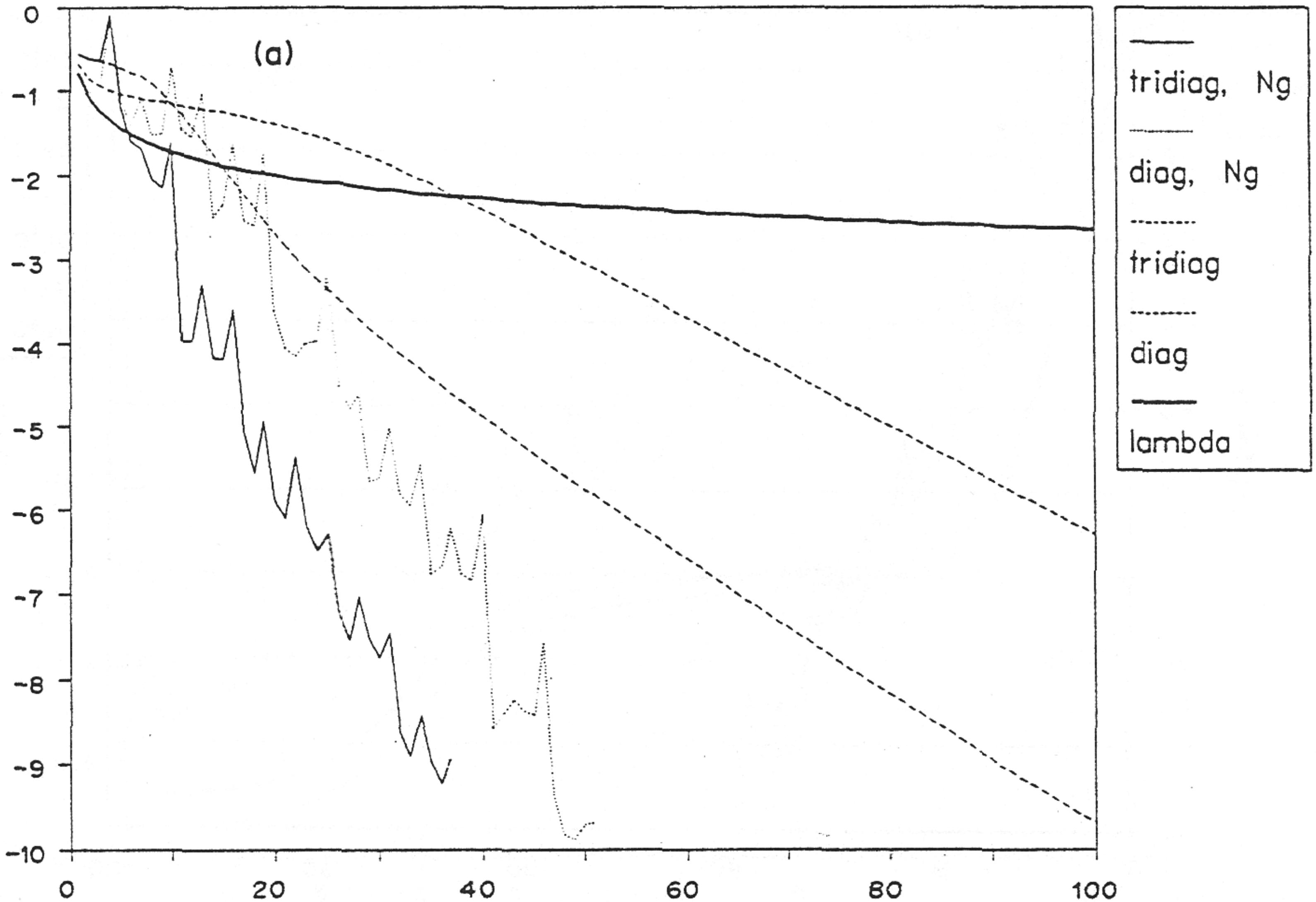
constructing Λ^*

- basic idea:
  - set $S = (0, \ldots, 1, 0, \ldots)$
  - perform FS
  - $\rightarrow$ delivers one column of Λ matrix
  - repeat for all spatial (radial) points
  - $\rightarrow$ compute Λ matrix
- if done like described $\rightarrow$ very expensive!
- however, it can be done analytically!

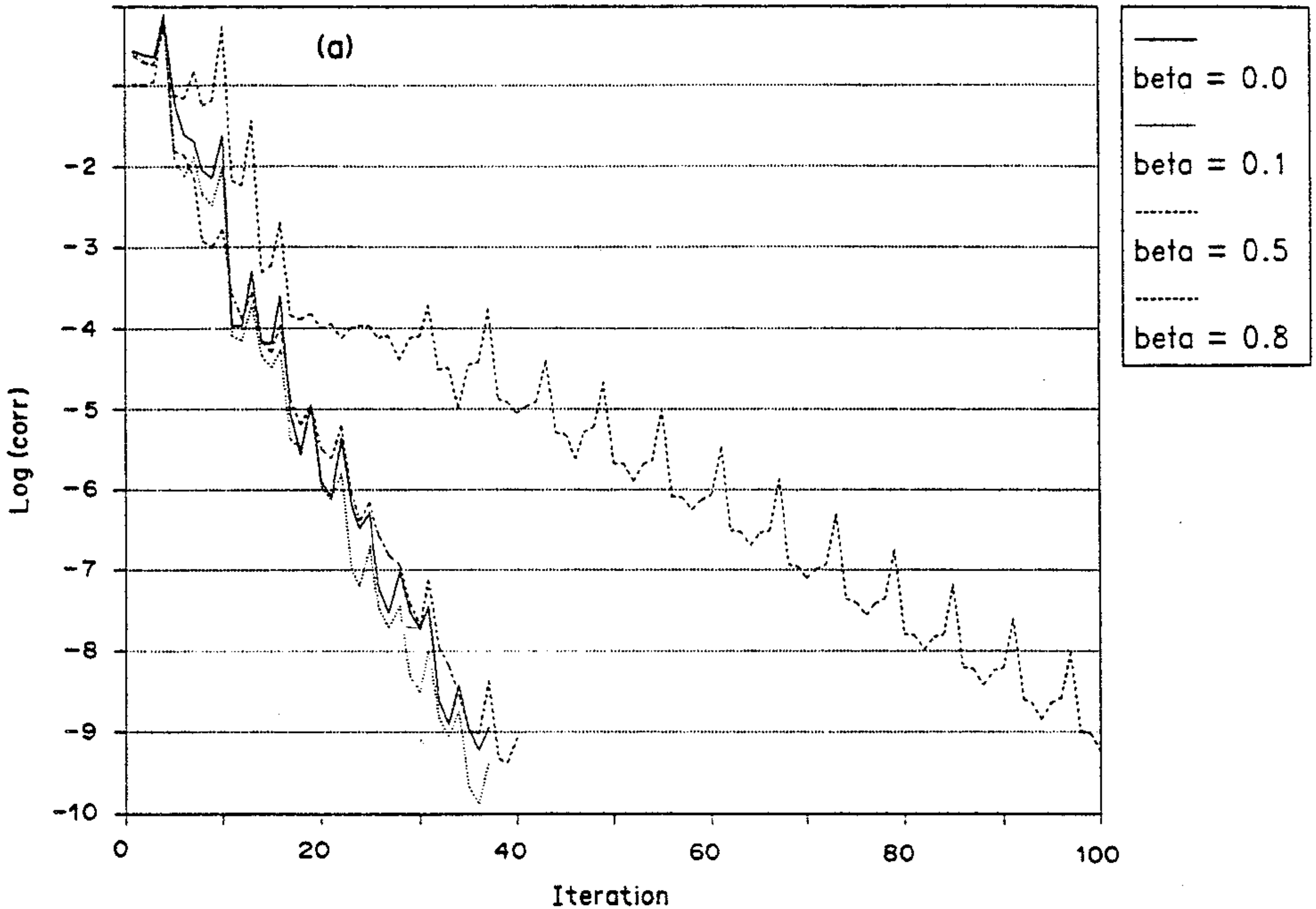
constructing Λ^*

- can be extended to full Λ matrix!
- this method can also be used in moving media
- can be extended to line transfer (later)

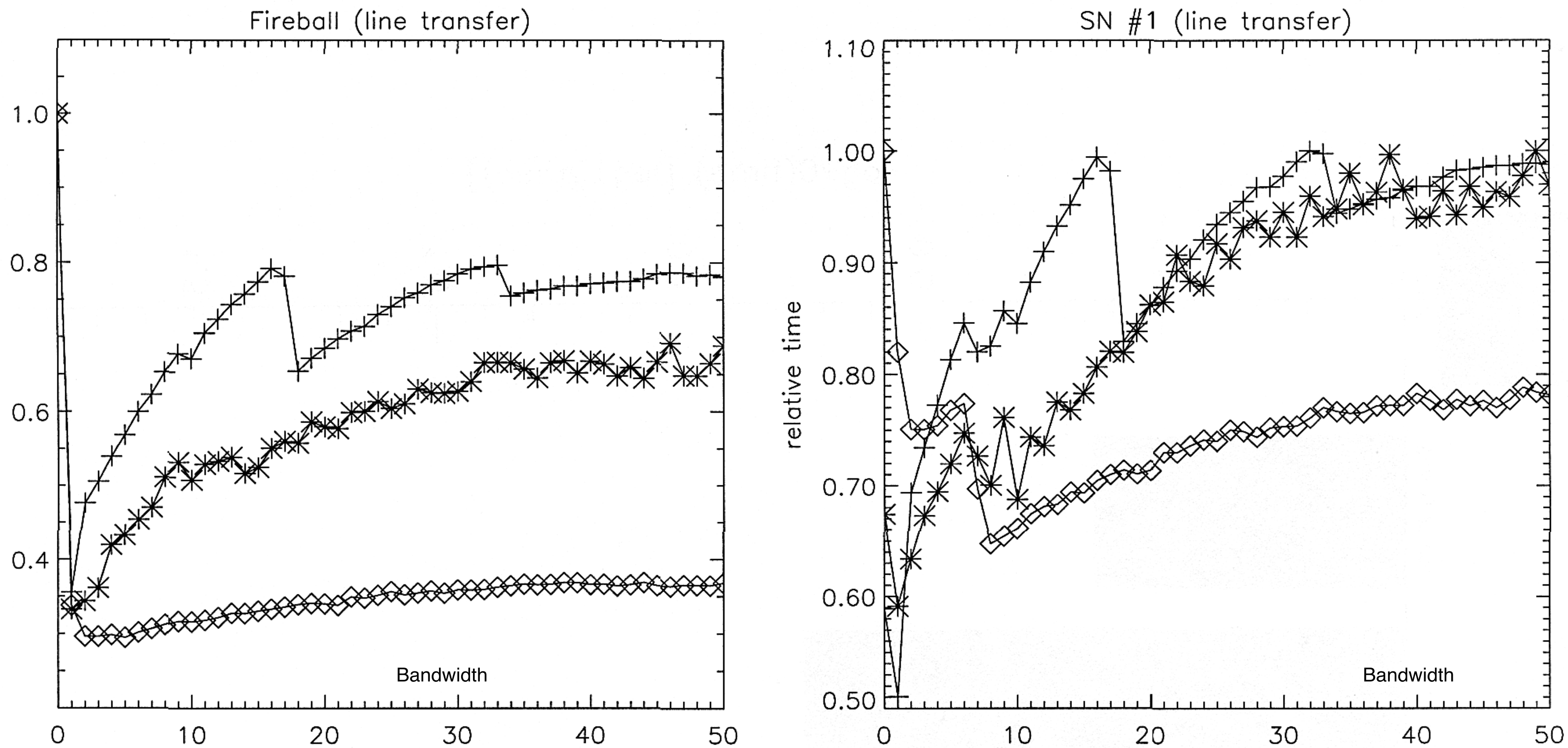
Convergence: static



Convergence: expanding

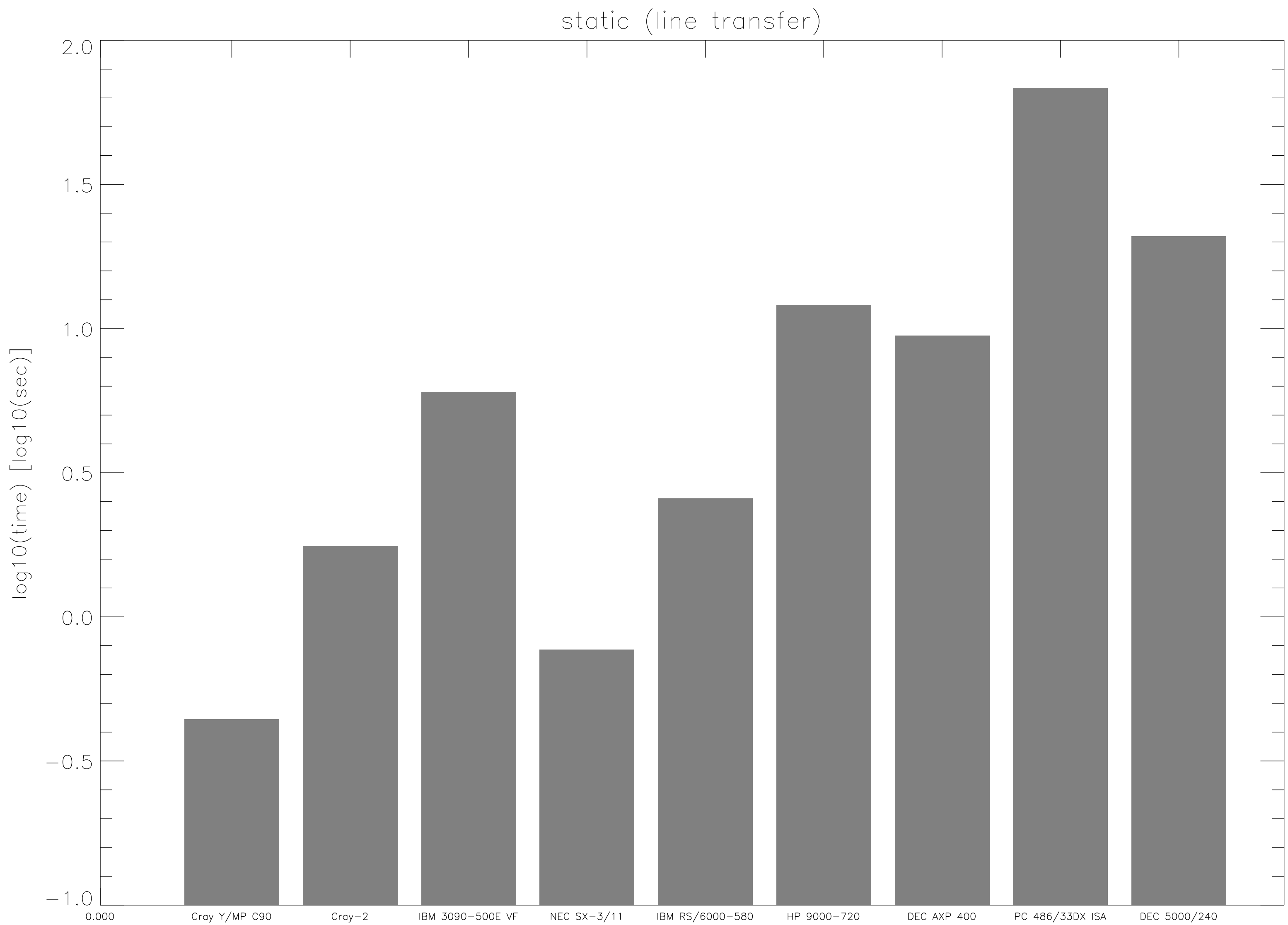


relative Performance



- SN: $v_{\max} = 0.13c$, extension 100
- Fireball: $v_{\max} = 0.9c$, extension 10^6
- $+$: Cray Y/MP C90, $*$: Cray-2,
  other: IBM 3090-500E VF

relative Performance



- historical ... ca. 1993